

1 Introduction

Both the matrix eigenvalue problem and the inverse eigenvalue problem are the subjects of intensive research [1,2]. This paper addresses both problems. A means of producing matrices with any specified eigenvalues and whose eigenvectors are known is presented. Both the elements of the matrix and the eigenvectors of the matrix are given in analytic form; this should facilitate the use of such matrices for analysis in other areas of linear algebra. A reversal of the matrix-generation procedure naturally suggests a matrix-diagonalization algorithm, and we derive the equation which needs to be solved to realize this. The findings we are going to present grew out of research into quantum foundations[3], but are here presented from a purely mathematical point of view.

The organization of this paper is as follows. In the next section, we present the basic result that makes possible the production of matrices with any desired eigenvalues. In Section 3, we outline the diagonalization routine that arises from it. In Section 4, we give the functions which for the case $N = 5$ enable the practical realization of both these projects. The results of preliminary application of this theory are reported in Section 5. We conclude with brief comments in Section 6.

2 Matrix Generation

Suppose that the N^2 functions $\phi(B_i; C_j)$ possess the orthonormality property

$$\sum_{l=1}^N \phi(B_l; C_i) \phi^*(B_l; C_j) = \delta_{ij}, \quad (1)$$

where B and C are parameters that take the values B_1, B_2, \dots, B_N and C_1, C_2, \dots, C_N respectively. The quantities $\phi(B_i; C_j)$ are functions of some variables, collectively denoted by x . If with the aid of the N arbitrary numbers $\lambda_1, \lambda_2, \dots, \lambda_N$ we form the $N \times N$ matrix $[M]$ using the rule

$$M_{ij} = \sum_{n=1}^N \phi^*(B_i; C_n) \phi(B_j; C_n) \lambda_n \quad (2)$$

we find that the eigenvectors of this matrix are

$$[\xi_i] = \begin{pmatrix} \phi^*(B_1; C_i) \\ \phi^*(B_2; C_i) \\ \vdots \\ \phi^*(B_N; C_i) \end{pmatrix} \quad (3)$$

with the respective eigenvalues λ_i . Indeed if we write

$$[V] = [M][\xi_i], \quad (4)$$

we find that the k th row of $[M][\xi_i]$ is

$$\begin{aligned} V_k &= \sum_{l=1}^N M_{kl} \phi^*(B_l; C_i) \\ &= \sum_{l=1}^N \left(\sum_{n=1}^N \phi^*(B_k; C_n) \phi(B_l; C_n) \lambda_n \right) \phi^*(B_l; C_i) \\ &= \sum_{n=1}^N \lambda_n \phi^*(B_k; C_n) \sum_{l=1}^N \phi(B_l; C_n) \phi^*(B_l; C_i) \\ &= \sum_{n=1}^N \lambda_n \phi^*(B_k; C_n) \delta_{ni} \\ &= \lambda_i \phi^*(B_k; C_i). \end{aligned} \quad (5)$$

Thus the eigenvalue equation

$$[M][\xi_i] = \lambda_i [\xi_i] \quad (6)$$

is satisfied. Property Eqn. (1) means that the eigenvectors are orthonormal:

$$\begin{aligned} [\xi_i]^\dagger [\xi_j] &= \sum_l \phi(B_l; C_i) \phi^*(B_l; C_j) \\ &= \delta_{ij}. \end{aligned} \quad (7)$$

These results give us a means of generating matrices with specified eigenvalues and known eigenvectors. We only need appropriate functions $\phi(B_i; C_j)$. An example of such functions will be given in Section 4. Given such functions, and a set of eigenvalues $\{\lambda_i\}$, we are able generate an infinite number of different matrices by varying the argument x of these functions. Each such matrix is generated together with its own orthonormal set of eigenvectors.

On the other hand, given a value of x , we can generate an infinite number of matrices by using different sets of eigenvalues. In this case, all these matrices share the same set of orthonormal eigenvectors.

3 A Diagonalization Algorithm

Since each matrix is generated by means of a specific value of the argument x , we should be able, at least in principle, to find a way of deducing this value of x for a given matrix. A procedure for doing this would constitute a diagonalization routine. In this section, we derive one such method.

Suppose we are given a matrix $[M]$ formed by the prescription Eqn. (2). Its eigenvectors are given by Eqn. (3). If we only knew the value of x that was used to form the matrix, then we could fix the eigenvectors and recover the eigenvalues. In order to develop a way of deducing the value of x , let us start, for the sake of convenience, by setting

$$\xi(C_i; B_j) = \phi^*(B_j; C_i) \quad (8)$$

so that

$$[\xi_i] = \begin{pmatrix} \phi^*(B_1; C_i) \\ \phi^*(B_2; C_i) \\ \vdots \\ \phi^*(B_N; C_i) \end{pmatrix} = \begin{pmatrix} \xi(C_i, B_1) \\ \xi(C_i, B_2) \\ \vdots \\ \xi(C_i, B_N) \end{pmatrix}. \quad (9)$$

Now, from

$$[M][\xi_i] = \lambda_i[\xi_i] \quad (10)$$

and the value of x that was used to produce the matrix, we can recover the eigenvalue λ_i from the r th row by using

$$(\lambda_i)_r = \frac{\sum_l M_{rl}\xi(C_i, B_l)}{\xi(C_i, B_r)}. \quad (11)$$

The s th row would give the same value. Hence, comparing the values from the r th and s th rows, we can write

$$(\lambda_i)_r - (\lambda_i)_s = 0. \quad (12)$$

This must be true for all possible combinations of r and s :

$$\sum_{r=1}^{N-1} \sum_{s=r+1}^N [(\lambda_i)_r - (\lambda_i)_s] = 0. \quad (13)$$

In other words:

$$\sum_{r=1}^{N-1} \sum_{s=r+1}^N \left[\frac{\sum_l M_{rl} \xi(C_i, B_l)}{\xi(C_i, B_r)} - \frac{\sum_l M_{sl} \xi(C_i, B_l)}{\xi(C_i, B_s)} \right] = 0. \quad (14)$$

But if the value of x used is not correct, the equation is not satisfied. Thus, this equation is a means of deducing the correct value of x . A form of this equation better adapted to numerical work is

$$\sum_{i=1}^N \sum_{r=1}^{N-1} \sum_{s=r+1}^N \left[\xi(C_i, B_s) \sum_l M_{rl} \xi(C_i, B_l) - \xi(C_i, B_r) \sum_l M_{sl} \xi(C_i, B_l) \right] = 0. \quad (15)$$

We have summed over all the eigenvalues by means of the index i because we must ensure that the equation is satisfied for every eigenvalue.

Once the correct value of x (denoted by x_0) is found, the eigenvalues are calculated using

$$(\lambda_i)_r = \left. \frac{\sum_l M_{rl} \xi(C_i, B_l)}{\xi(C_i, B_r)} \right|_{x_0} \quad (16)$$

and the eigenvectors are

$$[\xi_i] = \left(\begin{array}{c} \xi(C_i, B_1) \\ \xi(C_i, B_2) \\ \vdots \\ \xi(C_i, B_N) \end{array} \right) \Big|_{x_0}. \quad (17)$$

As is wise in the solution of such equations, it is necessary to establish that the value of x obtained is not spurious by putting it back into the equation. In this case, this means using it to calculate the eigenvalue using every line of the eigenvector. Thus, for each eigenvalues, there should be N values, which should agree in order for this value of x to be acceptable. Of course, in the case where one of the elements of the eigenvectors vanishes, it cannot be used to compute the eigenvalue, and there will be fewer than N values to compare.

4 Probability Amplitudes

In order to utilise the prescription for generating matrices with specified eigenvalues and to realise the algorithm for diagonalizing them, we need the

functions $\phi(B_i; C_j)$. Since the origins of the method presented in this paper lie in quantum theory, we shall call these functions probability amplitudes. A source for such functions is spin theory in quantum mechanics[3]. The treatment of a spin s system yields functions suitable for generating matrices of order $N = 2s + 1$. As an example, we take the spin-2 system. In the direction of the unit vector

$$\hat{\mathbf{c}} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \quad (18)$$

its operator is[4]

$$[\hat{\mathbf{c}} \cdot \mathbf{S}] = \begin{pmatrix} 2 \cos \theta & \sin \theta e^{-i\varphi} & 0 & 0 & 0 \\ \sin \theta e^{i\varphi} & \cos \theta & \frac{\sqrt{6}}{2} \sin \theta e^{-i\varphi} & 0 & 0 \\ 0 & \frac{\sqrt{6}}{2} \sin \theta e^{i\varphi} & 0 & -i \frac{\sqrt{6}}{2} \sin \theta e^{-i\varphi} & 0 \\ 0 & 0 & \frac{\sqrt{6}}{2} \sin \theta e^{i\varphi} & -\cos \theta & \sin \theta e^{-i\varphi} \\ 0 & 0 & 0 & \sin \theta e^{i\varphi} & -2 \cos \theta \end{pmatrix}. \quad (19)$$

The eigenvalues of this matrix are $2, 1, 0, -1$ and -2 , with the respective normalized eigenvectors

$$[\chi_{m=2}^{(\hat{\mathbf{c}})}] = \begin{pmatrix} \cos^4 \frac{\theta}{2} e^{-i2\varphi} \\ \sin \theta \cos^2 \frac{\theta}{2} e^{-i\varphi} \\ \frac{\sqrt{6}}{4} \sin^2 \theta \\ \sin \theta \sin^2 \frac{\theta}{2} e^{i\varphi} \\ \sin^4 \frac{\theta}{2} e^{i2\varphi} \end{pmatrix}, \quad (20)$$

$$[\chi_{m=1}^{(\hat{\mathbf{c}})}] = \begin{pmatrix} \sin \theta \cos^2 \frac{\theta}{2} e^{-i2\varphi} \\ (3 \sin^2 \frac{\theta}{2} - \cos^2 \frac{\theta}{2}) \cos^2 \frac{\theta}{2} e^{-i\varphi} \\ -\frac{\sqrt{6}}{2} \sin \theta \cos \theta \\ -(3 \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}) \sin^2 \frac{\theta}{2} e^{i\varphi} \\ -\sin \theta \sin^2 \frac{\theta}{2} e^{i2\varphi} \end{pmatrix}, \quad (21)$$

$$[\chi_{m=0}^{(\hat{\mathbf{c}})}] = \begin{pmatrix} \frac{\sqrt{6}}{4} \sin^2 \theta e^{-i2\varphi} \\ -\frac{\sqrt{6}}{2} \sin \theta \cos \theta e^{-i\varphi} \\ \frac{1}{2}(2 \cos^2 \theta - \sin^2 \theta) \\ \frac{\sqrt{6}}{2} \sin \theta \cos \theta e^{i\varphi} \\ \frac{\sqrt{6}}{4} \sin^2 \theta e^{i2\varphi} \end{pmatrix}, \quad (22)$$

$$[\chi_{m=-1}^{(\hat{\mathbf{c}})}] = \begin{pmatrix} \sin \theta \sin^2 \frac{\theta}{2} e^{-i2\varphi} \\ -(3 \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}) \sin^2 \frac{\theta}{2} e^{-i\varphi} \\ \frac{\sqrt{6}}{2} \sin \theta \cos \theta \\ (3 \sin^2 \frac{\theta}{2} - \cos^2 \frac{\theta}{2}) \cos^2 \frac{\theta}{2} e^{i\varphi} \\ -\sin \theta \cos^2 \frac{\theta}{2} e^{i2\varphi} \end{pmatrix} \quad (23)$$

and

$$[\chi_{m=-2}^{(\hat{\mathbf{c}})}] = \begin{pmatrix} \sin^4 \frac{\theta}{2} e^{-i2\varphi} \\ -\sin \theta \sin^2 \frac{\theta}{2} e^{-i\varphi} \\ \frac{\sqrt{6}}{4} \sin^2 \theta \\ -\sin \theta \cos^2 \frac{\theta}{2} e^{i\varphi} \\ \cos^4 \frac{\theta}{2} e^{i2\varphi} \end{pmatrix}. \quad (24)$$

We have used the index m to label the eigenvalues of the spin matrix because that is the convention. Also, we need to keep the distinction between the eigenvalues of this matrix and the eigenvalues $\{\lambda_i\}$ belonging to the matrix we wish to generate once we have obtained the probability amplitudes.

For the direction

$$\hat{\mathbf{b}} = (\sin \theta' \cos \varphi', \sin \theta' \sin \varphi', \cos \theta') \quad (25)$$

the eigenvectors of the operator are identical, except that (θ, φ) is replaced by (θ', φ') . We now associate the parameter B with the direction $\hat{\mathbf{b}}$, and the parameter C with the direction $\hat{\mathbf{c}}$. The notation is such that the values B_1, B_2, \dots, B_5 correspond respectively to the eigenvalues $2, 1, \dots, -2$. The same holds for C .

Some familiarity with spin theory in quantum mechanics is perhaps necessary to fully appreciate these functions. Each element in each eigenvector is a probability amplitude, and its modulus square is a probability for a certain measurement[3]. However, from a mathematical point of view, all that is needed is the fact that these elements possess the property Eqn. (1). The scalar products of the eigenvectors corresponding to $\hat{\mathbf{b}}$ with those corresponding to $\hat{\mathbf{c}}$ give the most generalized forms of these functions. Thus the required quantities are[5]

$$\phi(B_i; C_j) = [\chi_{m_j}^{(\hat{\mathbf{c}})}]^\dagger [\chi_{m_i}^{(\hat{\mathbf{b}})}]. \quad (26)$$

Evidently, in this case, $x = (\theta, \varphi, \theta', \varphi')$.

To illustrate the notation, we give a few of these functions:

$$\begin{aligned}
\phi(B_1; C_1) &= [\chi_{m=2}^{(\widehat{\mathbf{c}})}]^\dagger [\chi_{m=2}^{(\widehat{\mathbf{b}})}] \\
&= \cos^4 \frac{\theta}{2} \cos^4 \frac{\theta'}{2} e^{i2(\varphi-\varphi')} \\
&\quad + \sin \theta' \sin \theta \cos^2 \frac{\theta'}{2} \cos^2 \frac{\theta}{2} e^{i(\varphi-\varphi')} + \frac{3}{8} \sin^2 \theta' \sin^2 \theta \\
&\quad + \sin \theta' \sin \theta \sin^2 \frac{\theta'}{2} \sin^2 \frac{\theta}{2} e^{-i(\varphi-\varphi')} \\
&\quad + \sin^4 \frac{\theta}{2} \sin^4 \frac{\theta'}{2} e^{-i2(\varphi-\varphi')}, \tag{27}
\end{aligned}$$

$$\begin{aligned}
\phi(B_1; C_2) &= [\chi_{m=1}^{(\widehat{\mathbf{c}})}]^\dagger [\chi_{m=2}^{(\widehat{\mathbf{b}})}] \\
&= \sin \theta \cos^4 \frac{\theta'}{2} \cos^2 \frac{\theta}{2} e^{i2(\varphi-\varphi')} \\
&\quad + (3 \sin^2 \frac{\theta}{2} - \cos^2 \frac{\theta}{2}) \sin \theta' \cos^2 \frac{\theta'}{2} \cos^2 \frac{\theta}{2} e^{i(\varphi-\varphi')} \\
&\quad - \frac{3}{4} \sin^2 \theta' \sin \theta \cos \theta \\
&\quad - (3 \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}) \sin \theta' \sin^2 \frac{\theta'}{2} \sin^2 \frac{\theta}{2} e^{-i(\varphi-\varphi')} \\
&\quad - \sin \theta \sin^4 \frac{\theta'}{2} \sin^2 \frac{\theta}{2} e^{-i2(\varphi-\varphi')} \tag{28}
\end{aligned}$$

and

$$\begin{aligned}
\phi(B_5; C_5) &= [\chi_{m=-2}^{(\widehat{\mathbf{c}})}]^\dagger [\chi_{m=-2}^{(\widehat{\mathbf{b}})}] \\
&= \cos^4 \frac{\theta}{2} \cos^4 \frac{\theta'}{2} e^{i2(\varphi-\varphi')} \\
&\quad + \sin \theta' \sin \theta \sin^2 \frac{\theta'}{2} \sin^2 \frac{\theta}{2} e^{i(\varphi-\varphi')} + \frac{3}{8} \sin^2 \theta' \sin^2 \theta \\
&\quad + \sin \theta' \sin \theta \cos^2 \frac{\theta'}{2} \cos^2 \frac{\theta}{2} e^{-i(\varphi-\varphi')} \\
&\quad + \cos^4 \frac{\theta}{2} \cos^4 \frac{\theta'}{2} e^{-i2(\varphi-\varphi')}. \tag{29}
\end{aligned}$$

There are 22 other functions. We remind ourselves that since

$$\xi(C_i, B_j) = \phi^*(B_j; C_i), \tag{30}$$

then

$$\xi(C_i, B_j) = [\chi_{m_i}^{(\hat{\mathbf{b}})}]^\dagger [\chi_{m_j}^{(\hat{\mathbf{c}})}]. \quad (31)$$

These are the functions which enter into Eqn. (15).

In general, the probability amplitudes satisfy the inter-dependence law

$$\phi(B_i; C_j) = \sum_{l=1}^5 \phi(B_i; D_l) \phi^*(D_l; C_j), \quad (32)$$

where $\hat{\mathbf{d}}$ is a third direction to which corresponds the new parameter D . This is a particular form of a fundamental quantum property of three sets of probability amplitudes belonging to one quantum system [6],

$$\psi(A_i; C_j) = \sum_{l=1}^N \chi(A_i; B_l) \phi(B_l; C_j). \quad (33)$$

In the context of quantum theory, the quantity $|\phi(B_i; C_j)|^2$ is the probability that if the spin projection in the direction $\hat{\mathbf{b}}$ is $m_i \hbar$, a measurement of it in the direction $\hat{\mathbf{c}}$ yields the result $m_j \hbar$ [3]. Here \hbar is Planck's constant.

The parameters B and C , as observed, are defined by the angles (θ', φ') and (θ, φ) , and the indices of each parameter correspond to the eigenvalues. When $(\theta', \varphi') = (\theta, \varphi)$ in Eqn. (33), $A = C$ and Eqn. (1) is satisfied. Eqn. (1) is a special form of Eqn. (33). Within the context of quantum measurement theory, all this has a ready interpretation [6]. However, we stress that it is not necessary to understand this in order to use these functions to generate matrices customized as to eigenvalues and type.

With these probability amplitudes, the eigenvector for the eigenvalue λ_i is

$$[\xi_i] = \begin{pmatrix} \phi^*(B_1; C_i) \\ \phi^*(B_2; C_i) \\ \phi^*(B_3; C_i) \\ \phi^*(B_4; C_i) \\ \phi^*(B_5; C_i) \end{pmatrix} = \begin{pmatrix} [\chi_{m=2}^{(\hat{\mathbf{b}})}]^\dagger [\chi_{m_i}^{(\hat{\mathbf{c}})}] \\ [\chi_{m=1}^{(\hat{\mathbf{b}})}]^\dagger [\chi_{m_i}^{(\hat{\mathbf{c}})}] \\ [\chi_{m=0}^{(\hat{\mathbf{b}})}]^\dagger [\chi_{m_i}^{(\hat{\mathbf{c}})}] \\ [\chi_{m=-1}^{(\hat{\mathbf{b}})}]^\dagger [\chi_{m_i}^{(\hat{\mathbf{c}})}] \\ [\chi_{m=-2}^{(\hat{\mathbf{b}})}]^\dagger [\chi_{m_i}^{(\hat{\mathbf{c}})}] \end{pmatrix}. \quad (34)$$

Of course, since the matrix Eqn. (19) is a particular case of the form of Eqn. (2), its eigenvectors are given by Eqn. (34); evidently, they correspond to the angles $\theta' = 0$ and $\varphi' = 0$. In other words, for this case, the unit vector $\hat{\mathbf{b}}$ is in the z direction.

The association between the eigenvalues of the matrix we are generating and their eigenvectors is through Eqn. (2), where the value of C and the eigenvalue have the same index.

5 Results

A Fortran 77 program written to generate the matrices confirms that different families of matrices are obtained depending on the values of θ, φ, θ' and φ' , and on the character of the eigenvalues $\{\lambda_i\}$. The following observations are made, and can be predicted from the structure of the probability amplitudes and of Eqn. (2).

If all the angles are zero or if $\theta = \theta'$ and $\varphi = \varphi'$, the resulting matrix is diagonal.

If the eigenvalues are real, the matrix is Hermitian, but its elements depend on what values of the angles are used.

Whatever the eigenvalues, the matrix is symmetric if $\varphi = \varphi'$.

If $\varphi = \varphi'$, the eigenvectors are real.

If all the eigenvalues are pure imaginary, and the arguments are arbitrary, the matrix is anti-Hermitian.

If all the eigenvalues are pure imaginary, and $\varphi = \varphi'$, the matrix is imaginary but symmetric.

If the eigenvalues are arbitrary and the values of the arguments also arbitrary, the matrix is general.

The kinds of families that can be generated with different combinations of the eigenvectors and the arguments have not been fully investigated, but it seems probable that it does not require much ingenuity to generate such special forms as tridiagonal matrices, etc. Clearly, with this method, much of the inverse-eigenvalue problem is solved. Further investigation should show how to chose the values of the angles so as to obtain the desired matrices with specific values of certain elements.

As far as the diagonalization algorithm is concerned, it has only been partially validated at this time. The main complication here is that the non-linear equation (15) is in four variables. Therefore, solving it is somewhat complicated. The following was however achieved. A matrix was generated by means of Eqn. (2). In order to diagonalize it, three of the arguments were held at their actual values, and the remaining one was determined with the use of Eqn. (15). Which one of the variables to treat as unknown could be

decided at pleasure. The value of this variable was determined from Eqn. (15) by means of the bisection method. That this approach was successful indicates that once a fast and reliable method of solving a non-linear equation in four variables is employed, the algorithm will prove to be of some utility.

6 Conclusion and Discussion

In this paper, we have presented a prescription for forming matrices in such a way that their eigenvalues and eigenvectors are known. The method is very general indeed, and simply by varying the arguments, different kinds of matrices can be obtained. Always, the normalised eigenvectors are simultaneously given. The eigenvectors can be chosen to be real if desired, and the any combination of eigenvalues can be used. By essentially reversing the matrix-generation procedure, we have proposed an algorithm for diagonalizing matrices formed this way.

A good amount of work is still needed in order to fully understand and utilize the methods presented. For example, it is necessary to classify more properly and completely according to the values of the arguments θ, φ, θ' and φ' the kinds of matrices that can be generated. Such information would be particularly helpful in the solution of the non-linear equation. If a matrix is such that one or more of the arguments must have certain values, this makes so much easier the job of solving the equation, since the number of variables is reduced.

The matrices dealt with here are of order 5. In order to deal with matrices of higher order, it is necessary to have the functions $\{\phi\}$ for those cases. A source of these functions will always be spin theory, but treatment of other N -dimensional quantum systems should produce other forms of the functions. Each such set of functions probably produces matrices of different characters. As such, it expands the range of uses to which we can put these matrices.

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7 References

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